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Non-Archimedean Integration and Strict Topologies

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Introduction

Let $C_b(X, E)$ be the space of all bounded continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. By $C_{rc}(X, E)$ we denote the space of all $f \in C_b(X, E)$ for which $f(X)$ is a relatively compact subset of E. In section 2 of this paper we show that, if E is polar and complete and Y a closed subset of X which is either compact or X is ultranormal, then there exists a linear map $T : C_{rc}(Y, E) \to C_{rc}(X, E)$ such that Tf is an extension of f and $\|Tf\|_p = \|f\|_p$ for all $f \in C_{rc}(Y, E)$ and every polar continuous seminorm p on E. Using this we identify in section 3 the completion of the space $C_b(X, E)$ under the strict topology β_o when E is polar. If $K(X)$ is the algebra of all clopen (i.e. both closed and open) subsets of X, we define in section 4 the product of certain $\mathbb{K}$-valued finitely-additive measures on $K(X)$ with E'-valued measures on $K(Y)$, where Y is another zero-dimensional topological space. Finally in sections 5 and 6 we define the so called (VR)-integral and Q-integral of functions in E^X with respect to certain measures on $K(X)$.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ stands for a complete non-Archimedean valued field whose valuation is non-trivial. By a seminorm, on a vector space E over $\mathbb{K}$, we mean a non-Archimedean seminorm. Similarly, by a locally convex space we mean a non-Archimedean locally convex space over $\mathbb{K}$. For E a locally convex space, we denote by $cs(E)$ the collection of all continuous seminorms on E, by E' its dual space and by $\hat{E}$ its completion. If F is another locally convex space, then $E \otimes F$ will be the tensor product of E, F with the projective topology.

Let now X be a zero-dimensional Hausdorff topological space and E a Hausdorff locally convex space. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [4]) and by $\upsilon_o X$ the $\mathbb{N}$-repletion of X ($\mathbb{N}$ is the set of natural numbers), i.e. the subspace of $\beta_o X$ consisting of all $x \in \beta_o X$ with the following property: For each sequence (V_n) of neighborhoods of x in $\beta_o X$ we have that $\bigcap V_n \cap X \neq \emptyset$. The space X is called $\mathbb{N}$-replete if $X = \upsilon_o X$. We will denote by $C_b(X, E)$ the space of all bounded continuous E-valued functions on X and by $C_{rc}(X, E)$ the space of all $f \in C_b(X, E)$ for which $f(X)$ is relatively compact in E. In case $E = \mathbb{K}$, we will simply write $C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the $\mathbb{K}$-valued characteristic function of A in X and by $\overline{A}^{\beta_o X}$ the closure of A in $\beta_o X$. Every $f \in C_{rc}(X, E)$ has a unique continuous extension f^{β_o} to all of $\beta_o X$. For f an E-valued function on X, p a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology β_o on $C_b(X, E)$ (see [7]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and h is in the space $B_o(X)$ of all bounded $\mathbb{K}$-valued functions on X which vanish at infinity, i.e. for each $\epsilon > 0$ there exists a compact subset Y of X such that $|h(x)| < \epsilon$ if x is not in Y. Let Ω be the family of all compact subsets of $\beta_o X$ which are disjoint from X. For $H \in \Omega$, let C_H be the space of all $h \in C_{rc}(X)$ whose continuous extension h^{β_o} vanishes on H. For $p \in cs(E)$, let $\beta_{H,p}$ be the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, h \in C_H$. The inductive limit of the topologies $\beta_{H,p}$, as H ranges over Ω, is denoted by β_p while β is the projective limit of the topologies $\beta_p, p \in cs(E)$. The following Theorem is proved in [11].

Theorem 1.1 *An absolutely convex subset V of $C_b(X, E)$ is a $\beta_{H,p}$-neighborhood of zero iff the following condition is satisfied: For each $r > 0$, there exist $\epsilon > 0$ and a clopen subset A of X, with $\bar{A}^{\beta_o X} \cap H = \emptyset$, such that*

$$\{f \in C_b(X, E) : \|f\|_p \leq r, \|f\|_{A,p} \leq \epsilon\} \subset V.$$

Let now $K(X)$ be the algebra of all clopen, (i.e. closed and open) subsets of X. We denote by $M(X, E')$ (see [6]) the space of all finitely-additive E'-valued measures m on $K(X)$ for which $m(K(X))$ is an equicontinuous subset of E'. For each m in $M(X, E')$ there exists $p \in cs(E)$ with $m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0, A \supset B \in K(X)\}.$$

The space of all $m \in M(X, E')$ with $m_p(X) < \infty$ is denoted by $M_p(X, E')$. We denote by $M_\tau(X, E')$ the space of all $m \in M(X, E')$ such that, for every decreasing net (A_δ) of clopen subsets of X, with $\cap A_\delta = \emptyset$, there exists $p \in cs(E)$ such that $m_p(A_\delta) \to 0$. Also by $\mathcal{M}_{\tau,p}(X, E')$ we denote the space of all $m \in M_p(X, E')$ such that $m_p(A_\delta) \to 0$ for every decreasing net (A_δ) of clopen subsets of X with $\cap A_\delta = \emptyset$. Let

$$\mathcal{M}_\tau(X, E') = \bigcup_{p \in cs(E)} \mathcal{M}_{\tau,p}(X, E').$$

For $p \in cs(E)$, we denote by $M_{t,p}(X, E')$ the space of all $m \in M_p(X, E')$ for which m_p is tight, i.e. for every $\epsilon > 0$, there exists a compact subset Y of X such that